
LINEAR AND NONLINEAR ANALYSIS OF THERMAL INSTABILITY IN A POROUS SATURATED BY A NANOFLUID

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Abstract

In this present article, the onset of convection in a horizontal layer of a porous medium saturated by a nanofluid is investigated analytically using linear and weakly nonlinear analysis. The model used for the nanofluid incorporates the effect of Brownian motion and thermophoresis. The effect of Raleigh-Darcy number, Lewis Number, modified diffusivity ratio, Vadasz number and normalized porosity parameter on the stability of the system is investigated. The analysis reveals that for a typical nanofluid (with large Lewis number) the prime effect of the nanofluids is via buoyancy effect coupled with the conservation of nanoparticles to the thermal energy equation being a second –order effect. Stationary and oscillatory modes of convections have been studied. It is found that the critical thermal Raleigh number can be reduced or increased by a substantial amount, depending on whether the basic nanoparticle distribution is top-heavy, by the presence of the nanoparticles. Oscillatory instability is possible in the case of a bottom-heavy nanoparticle distribution. The linear stability analysis is based on normal mode technique, while for nonlinear theory is based on the truncated representation of Fourier series method. A weakly nonlinear analysis is used to obtain the concentration and thermal Nusselt number. The behavior of the concentration and thermal Nusselt numbers is investigated by a solving the finite amplitude equations. Obtained results have been presented graphically and discussed in details.

Keywords:

Nanofluid, Porous medium,
Instability, Natural
convection.

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1. Introduction

The term "nanofluid" refers to a liquid containing a suspension of submicron solid particles (nanoparticles). The term was coined by Choi [1]. The characteristic feature of nanofluids is thermal conductivity enhancement, a phenomenon observed by Masuda et al. [2]. This phenomenon suggests the possibility of using nanofluids in advanced nuclear systems (Buongiorno and Hu [25]). Nanofluids are mixtures of base fluid such as water or ethylene-glycol with a very small amount of nanoparticles such as metallic or metallic oxide particles (Cu, CuO, Al₂O₃), having dimensions from 1 to 100 nm. Buongiorno [5] conducted a comprehensive study to account for the unusual behavior of nanofluids based on Inertia, Brownian diffusion, thermophoresis, diffusiophoresis, Magnus effects, fluid drainage and gravity settling, and proposed a model incorporating the effects of Brownian diffusion and the thermophoresis. Studies pertaining to thermal conductivity enhancement by nanofluid have been conducted by Eastman et al [3], Das et al [4] and others. They claimed a 10-30% increase in thermal conductivity by using very low concentrations of nanofluid.

Due to applications of nanofluids and porous media theory in drying, freezing of foods and applications in every day technology such as microwave heating, rapid heat transfer from computer chips via use of porous metal form and their use in heat pipes, One of the most significant scientific challenges in the industrial area is cooling, which applies to many diverse productions including microelectronics, transportation and manufacturing. Technological developments such as microelectronic devices operating at high speeds, high power engines, a brighter optical devices and driving increases thermal loads, requiring advances in cooling .

There are several studies available in which phenomena related to the onset of convectional in a porous medium have been investigated. Few of them are Parlstein [6] ,Chakrabarti and Gupta [7], Patil and Vaidyanathan [8], Vadasz [9] convection in porous medium has been studied by many authors including Horton and Roger [10], Lapwood [11], Nield [12], Rudraiah and Malshetty [13], Murray and Chen [14], Bhadauria [15], Vafai [16], Neild and Bejan [17].

Recently, Nield and Kuznetsov [18, 19] for the Darcy Model, Kuznetsov and Nield [20] also studied local thermal non-equilibrium and flow past vertical plate for nanofluids. Agarwal et al [20], Bhadauria et al [22] studied the same problem. We study the linear and nonlinear analysis of thermal instability in a porous layer saturated by nanofluids in this present article.

2. Conservation Equation for a Nanofluid

First, we outline the derivation of conservation equations applicable to a nanofluid in the absence of a solid matrix. Later we modify these equations to the case of a porous medium saturated by the nanofluid. The Buongiorono model treats the nanofluid as a two components mixture (base fluid plus nanoparticles) with the following assumptions.

1. Incompressible flow
2. No chemical reaction
3. Negligible external forces
4. Dilute mixture
5. Negligible viscous dissipation
6. Negligible radiative heat transfer
7. Nanoparticles and base fluid locally in thermal equilibrium.

In sections 2 and 3, all the variables are dimensional. The continuity equation for the nanofluid is

$$\nabla \cdot \mathbf{v} = 0 \tag{1}$$

Here \mathbf{v} is the nanofluid velocity.

The conservation equation for the nanoparticles in the absence of chemical reactions is

$$\frac{\partial \phi}{\partial t} + \mathbf{v} \cdot \nabla \phi = -\frac{1}{\rho_p} \nabla \cdot \mathbf{j}_p \quad (2)$$

Where ϕ is nanoparticle volume fraction, ρ_p is the nanoparticle mass density and \mathbf{j}_p is the diffusion mass flux for the nanoparticles, given as the sum of two diffusion terms (Brownian diffusion and thermophoresis) by

$$\mathbf{j}_p = \mathbf{j}_{p,B} + \mathbf{j}_{p,T} = -\rho_p D_B \nabla \phi - \rho_p D_T \frac{\nabla T}{T} \quad (3)$$

(Thermophoresis is the "particle" equivalent of the Soret effect in gaseous or liquid mixtures)

Here D_B is the Brownian diffusion coefficient given by the Einstein-stokes Equation

$$D_B = \frac{k_B T}{3\pi\mu d_p} \quad (4)$$

Where k_B Boltzmann's constant, μ is the viscosity of the fluid and d_p is the nanoparticle diameter. Use has been made of the expression

$$\mathbf{V}_T = \tilde{\beta} \frac{\mu}{\rho} \frac{\nabla T}{T} \quad (5)$$

For the thermophoretic velocity \mathbf{V}_T here ρ is the fluid density and the proportionality factor $\tilde{\beta}$ is given by

$$\tilde{\beta} = 0.26 \frac{k}{2k + k_p} \quad (6)$$

Where k and k_p are the thermal conductivities of the fluid and the particle material. Hence the thermophoretic diffusion flux is given by

$$\mathbf{j}_{p,T} = \rho_p \phi \mathbf{V}_T = -\rho_p D_T \frac{\nabla T}{T} \quad (7)$$

Where the thermophoretic diffusion coefficient is given by

$$D_T = \tilde{\beta} \frac{\mu}{\rho} \phi \quad (8)$$

Eq. (2) and (3) then produce the conservation equation in the form

$$\frac{\partial \phi}{\partial t} + \mathbf{v} \cdot \nabla \phi = \nabla \cdot \left(D_B \nabla \phi + D_T \frac{\nabla T}{T} \right) \quad (9)$$

The momentum equation for a nanofluid takes the same form as for a pure fluid, but it should be remembered that μ is strong function of ϕ . If one introduces a buoyancy force and adopts the Boussinesq approximation, then the momentum equation can be written as

$$\rho \left(\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} \right) = -\nabla \phi + \mu \nabla^2 \mathbf{v} + \rho \mathbf{g}, \quad (10)$$

Where

$$\rho = \phi \rho_p + (1 - \phi) \rho_f \quad (11)$$

The nanofluid density ρ can be approximated by the base-fluid density ρ_f when ϕ is small. Then when the Boussinesq approximation is adopted the buoyancy term is approximated by

$$\rho \mathbf{g} \cong [\phi \rho_p + (1 - \phi) \rho_f \{1 - \beta(T - T_0)\}] \mathbf{g} \quad (12)$$

The thermal energy equation for a nanofluid can be written as

$$\rho c \left(\frac{\partial T}{\partial t} + \mathbf{v} \cdot \nabla T \right) = -\nabla \cdot \mathbf{q} + h_p \nabla \cdot \mathbf{j}_p \quad (13)$$

where c is the nanofluid specific heat, T is the nanofluid temperature, h_p is the specific enthalpy of the nanoparticle material and q is the energy flux, relative to a frame moving with the nanofluid velocity \mathbf{v} , given by

$$\mathbf{q} = -k\nabla T + h_p \mathbf{j}_p, \tag{14}$$

where k is the nanofluid thermal conductivity. Substituting Eq. (14) in Eq (13) yields

$$\rho c \left(\frac{\partial T}{\partial t} + \mathbf{v} \cdot \nabla T \right) = \nabla \cdot (k \nabla T) - c_p \mathbf{j}_p \cdot \nabla T, \tag{15}$$

In deriving this equation use has been made of a vector identity and the fact (deriving from assumption (7)) that $\nabla h_p = c_p \nabla T$, where c_p is the nanoparticle specific heat of the material constituting the nanoparticles while c is the specific heat (at constant pressure) of the fluid. Then substitution of Eq (3) in Eq (15) gives the final form

$$\rho c \left(\frac{\partial T}{\partial t} + \mathbf{v} \cdot \nabla T \right) = \nabla \cdot (k \nabla T) + \rho_p c_p \left(D_B \nabla \phi \cdot \nabla T + D_T \frac{\nabla T \cdot \nabla T}{T} \right) \tag{16}$$

3. Conservation Equation for a Porous Medium Saturated by a Nanofluid

We consider a porous medium whose porosity is denoted by ε and permeability by K . A subscript s will now be used to denote properties of the solid matrix. The Darcy velocity is denoted by v_D . This is related to v by $v_D = \varepsilon v$. we now have to deal with the following four field equation (corresponding to Eq (1), (10), (16), (9) for total mass, momentum, thermal energy and nanoparticle, respectively

$$\nabla \cdot \mathbf{v}_D = 0 \tag{17}$$

$$\rho \left(\frac{1}{\varepsilon} \frac{\partial \mathbf{v}_D}{\partial t} + \frac{1}{\varepsilon^2} \mathbf{v}_D \cdot \nabla \mathbf{v}_D \right) = -\nabla p + \tilde{\mu} \nabla^2 \mathbf{v}_D - \frac{\mu}{K} \mathbf{v}_D + \left[\phi \rho_p + (1-\phi) \left\{ \rho (1 - \beta (T - T_0)) \right\} \right] \mathbf{g} \tag{18}$$

$$(\rho c)_m \frac{\partial T}{\partial t} + (\rho c)_f \mathbf{v}_D \cdot \nabla T = \nabla \cdot (k_m \nabla T) + \varepsilon (\rho c)_p \left[D_B \nabla \phi \cdot \nabla T + D_T \frac{\nabla T \cdot \nabla T}{T} \right] \tag{19}$$

$$\frac{\partial \phi}{\partial t} + \frac{1}{\varepsilon} \mathbf{v}_D \cdot \nabla \phi = \nabla \cdot \left[D_B \nabla \phi + D_T \frac{\nabla T}{T} \right] \tag{20}$$

Here we have introduced the effect of viscosity $\tilde{\mu}$, the effective heat capacity $(\rho c)_m$ and the effective thermal conductivity k_m of the porous medium.

In deriving Eq (17)-(20) we have assumed that the Brownian motion and thermophoresis processes remain coherent while volume averages over a representative elementary volume are taken. This assumption can be questioned. In the context of modeling transport in porous media, Eq (17) and (18) are standard. Eq (20) involves just intrinsic quantities in the sense that the average is being taken over the nanofluid only and the solid matrix is not involved. The question thus reduces to whether the terms within the square brackets on the right-hand side of Eq (19) need modification. We recall that in nanofluids the particles are so small that for practical purposes they remain in suspension in a uniform manner. We emphasize our assumption that the nanoparticles are suspended in nanofluid using either surfactant or surface charge Technology, something that prevents particles from agglomeration and deposition on the porous matrix. We suggest that then it is reasonable to assume as a first approximation that no modification to Eq (19) is necessary.

4. Application to the Horton–Rogers–Lapwood Problem

We select a coordinate frame in which the z-axis is aligned vertically upwards. We consider a horizontal layer of a porous medium confined between the planes $z^* = 0$ and $z^* = H$. From now on asterisks are used to denote dimensional variables (previously an asterisk has not been needed because all the variables were dimensional). Each boundary wall is assumed to be impermeable and perfectly thermally conducting. The temperatures at the lower and upper wall are taken to be T_h^* and T_c^* , the former being the greater. For simplicity, Darcy's law is assumed to hold and the Oberbeck–Boussinesq approximation is employed. Homogeneity and local thermal equilibrium in the porous medium are assumed. The reference temperature is taken to be T_c^* in the linear theory being applied here the temperature change in the fluid is assumed to be small in comparison with T_c^* . Eq (18)–(20) takes the form

$$\frac{\rho}{\varepsilon} \frac{\partial \mathbf{v}^*}{\partial t} = -\nabla^* p^* - \frac{\mu}{K} \mathbf{v}_D^* + \left[\phi^* \rho_p + (1 - \phi^*) \left\{ \rho (1 - \beta (T^* - T_c^*)) \right\} \right] \mathbf{g} \quad (21)$$

$$(\rho c)_m \frac{\partial T^*}{\partial t} + (\rho c)_f \mathbf{v}_D^* \cdot \nabla T = (k_m \nabla^2 T^*) + \varepsilon (\rho c)_p \left[D_B \nabla^* \phi^* \cdot \nabla^* T^* + (D_T / T_c^*) \nabla^* T^* \cdot \nabla^* T^* \right]$$

(22)

$$\frac{\partial \phi^*}{\partial t} + \frac{1}{\varepsilon} \mathbf{v}_D^* \cdot \nabla^* \phi^* = D_B \nabla^2 \phi^* + (D_T / T_c^*) \nabla^2 T^* \quad (23)$$

We write $\mathbf{v}_D^* = (u^*, v^*, w^*)$

We assume that the temperature and the volumetric fraction of the nanoparticles are constant on the boundaries. Thus the boundary conditions are

$$w^* = 0, T^* = T_h^*, \phi^* = \phi_0^* \text{ at } z^* = 0 \quad (24)$$

$$w^* = 0, T^* = T_c^*, \phi^* = \phi_1^* \text{ at } z^* = H \quad (25)$$

We recognize that our choice of boundary conditions imposed on ϕ^* is somewhat arbitrary. It could be argued that zero particle flux on the boundaries is more realistic physically, but then one is faced with the problem that it appears that no steady-state solution for the basic conduction equations is then possible, so that in order to make analytical progress it is necessary to freeze the basic profile for ϕ^* , and at that stage our choice of boundary conditions is seen to be quite realistic.

We introduce dimensionless variables as follows. We define

$$(x, y, z) = (x^*, y^*, z^*) / H, \quad t = t^* \alpha_m / \sigma H^2,$$

$$(u, v, w) = (u^*, v^*, w^*) H / \alpha_m, \quad p = p^* K / \mu \alpha_m$$

$$\phi = \frac{\phi^* - \phi_0^*}{\phi_1^* - \phi_0^*}, \quad T = \frac{T^* - T_c^*}{T_h^* - T_c^*}, \quad (26)$$

Where

$$\alpha_m = \frac{k_m}{(\rho c)_f}, \quad \sigma = \frac{(\rho c)_m}{(\rho c)_f}, \quad (27)$$

Then Eq (17) and (21) - (25) take the form

$$\nabla \cdot \mathbf{v} = 0 \quad (28)$$

$$\gamma_a \frac{\partial v}{\partial t} = -\nabla p - \mathbf{v} - Rm \hat{e}_z + RaT \hat{e}_z - Rn \phi \hat{e}_z, \quad (29)$$

$$\frac{\partial T}{\partial t} + \mathbf{v} \cdot \nabla T = \nabla^2 T + \frac{N_B}{Le} \nabla \phi \cdot \nabla T + \frac{N_A N_B}{Le} \nabla T \cdot \nabla T \quad (30)$$

$$\frac{1}{\sigma} \frac{\partial \phi}{\partial t} + \frac{1}{\varepsilon} \mathbf{v} \cdot \nabla \phi = \frac{1}{Le} \nabla^2 \phi + \frac{N_A}{Le} \nabla^2 T \tag{31}$$

$$w = 0, T = 1, \phi = 0 \text{ at } z = 0 \tag{32}$$

$$w = 0, T = 0, \phi = 1 \text{ at } z = 1 \tag{33}$$

here

$$\gamma_a = \frac{\eta}{\nu_a}, Le = \frac{\alpha_m}{D_B}, Ra_T = \frac{\rho g \beta K H (T_h^* - T_c^*)}{\mu \alpha_m}, Rm = \frac{[\rho_p \phi_1^* + \rho(1 - \phi_1^*)] g K H}{\mu \alpha_m}, Rn = \frac{(\rho_p - \rho)(\phi_1^* - \phi_0^*) g K H}{\mu \alpha_m}$$

$$, N_A = \frac{D_T (T_h^* - T_c^*)}{D_B T_c^* (\phi_1^* - \phi_0^*)}, N_B = \frac{\varepsilon (\rho c)_p}{(\rho c)_f} (\phi_1^* - \phi_0^*),$$

The parameter Le is a Lewis number and Ra_T is the familiar thermal Rayleigh–Darcy number. The new parameters Rm and Rn may be regarded as a basic-density Rayleigh number and a concentration Rayleigh number respectively and Vadasz number $Va = \frac{\varepsilon^2 Pr}{Da}$ ($Pr = \frac{\nu}{k_{11}}$, is the

Prandtl number, $Da = \frac{K}{d^2}$ is the Darcy number) and $\eta = \frac{\varepsilon}{\gamma}$ the normalized porosity. The parameter

N_A is a modified diffusivity ratio and is somewhat similar to the Sort parameter that arises in cross-diffusion phenomena in solutions, while N_B is a modified particle-density increment. In the spirit of the Oberbeck–Boussinesq approximation, Eq (29) has been linearized by the neglect of a term proportional to the product of ϕ and T . This assumption is likely to be valid in the case of small temperature gradients in a dilute suspension of nanoparticles.

4.1. Basic solution

We seek a time-independent quiescent solution of Eq (28)–(33) with temperature and nanoparticle volume fraction varying in the z -direction only that is a solution of the form

$$\mathbf{v} = 0, T = T_b(z), \phi = \phi_b(z)$$

Eq (30) and (31) reduce to

$$\frac{d^2 T_b}{dz^2} + \frac{N_B}{Le} \frac{d\phi_b}{dz} \frac{dT_b}{dz} + \frac{N_A N_B}{Le} \left(\frac{dT_b}{dz} \right)^2 = 0 \tag{34}$$

$$\frac{d^2 \phi_b}{dz^2} + N_A \frac{dT_b}{dz} = 0 \tag{35}$$

Using the boundary condition (32) and (33) Eq (41) maybe integrated to give

$$\phi_b = -N_A T_b + (1 - N_A)Z + N_A \tag{36}$$

and substitution of this into Eq (34) gives

$$\frac{d^2 T_b}{dZ^2} + \frac{(1 - N_A)N_B}{Le} \frac{dT_b}{dZ} = 0 \tag{37}$$

The solution of Eq (37) satisfying Eq (32) and (33) is

$$T_b = \frac{1 - e^{-(1 - N_A)N_B(1 - Z)/Le}}{1 - e^{-(1 - N_A)N_B/Le}} \tag{38}$$

The remainder of the basic solution is easily obtained by first substituting in Eq (42) to obtain ϕ_b and then using integration of Eq (29) to obtain P_b .

According to Buongiorno [5] for most nanofluids investigated so far for $Le/(\phi_1^* - \phi_0^*)$ is large of order $10^5 - 10^6$ and since the nanoparticle fraction decrement is typically no smaller than 10^3

this means so that Le is large of order $10^2 - 10^3$ while N_A is no greater than about 10. Then the exponents in Eq (37) and (38) are small and so to a good approximation one has

$$T_b = 1 - Z \tag{39}$$

and so

$$\phi_b = z \tag{40}$$

4.2. Perturbation solution

We now superimpose perturbations on the basic solution. We write

$$V = V' , p = p_p + p' , T_b = T_b + T' , \phi = \phi_b + \phi' \tag{41}$$

Substitute in Eq (28)–(33), and linearize by neglecting products of primed quantities. The following equations are obtained when Eq (39) and (40) are used.

$$\nabla V' = 0 \tag{42}$$

$$\gamma_a \frac{\partial \mathbf{v}'}{\partial t} = -\nabla p' - \mathbf{v}' + RaT\hat{e}_z - Rn\phi'\hat{e}_z; \tag{43}$$

$$\frac{\partial T'}{\partial t} - w' = \nabla^2 T' + \frac{N_B}{Le} \left(\frac{\partial T'}{\partial z} - \frac{\partial \phi'}{\partial z} \right) - \frac{2N_A N_B}{Le} \frac{\partial T'}{\partial z}; \tag{44}$$

$$\frac{1}{\sigma} \frac{\partial \phi'}{\partial t} + \frac{1}{\varepsilon} w' = \frac{1}{Le} \nabla^2 \phi' + \frac{N_A}{Le} \nabla^2 T' \tag{45}$$

$$w' = 0, T' = 0, \phi' = 0 \text{ at } Z = 0 \text{ and at } z = 1 \tag{46}$$

It will be noted that the parameter Rm is not involved in these and subsequent equations. It is just a measure of the basic static pressure gradient.

For the case of a regular fluid (not a nanofluid) the parameters Rn , N_A and N_B are zero, the second term in Eq (45) is absent because $d\phi_b/dz = 0$ and then Eq (45) is satisfied trivially. The remaining equations are reduced to the familiar equations for the Horton–Roger–Lapwood problem.

The six unknowns $u', v', w', p', T', \phi'$ can be reduced to three by operating on Eq (43) with $\hat{e}_z \text{ curl curl}$ and using the identity $\text{curl curl} \equiv \text{grad div} - \nabla^2$ together with Eq (42).

The result is

$$\nabla^2 w' + \gamma_a s \nabla^2 w' = Ra_T \nabla_H^2 T' - Rn \nabla_H^2 \phi' \tag{47}$$

Here ∇_H^2 is the two-dimensional Laplacian operator on the horizontal plane.

The differential Eq. (47), (44), (45) and the boundary conditions (46) constitute a linear boundary-value problem that can be solved using the method of normal modes.

We write

$$(w', T', \phi') = [W(z), \Theta(z), \Phi(z)] \exp(st + ilx + imy) \tag{48}$$

and substitute into the differential equations to obtain

$$(1 + \gamma s)(D^2 - \alpha^2)W + Ra_T \alpha^2 \Theta - Rn \alpha^2 \Phi = 0 \tag{49}$$

$$W + \left(D^2 + \frac{N_A}{Le} D - \frac{2N_A N_B}{Le} D - \alpha^2 - s \right) \Theta - \frac{N_B}{Le} D \Phi = 0 \tag{50}$$

$$\frac{1}{\varepsilon} W - \frac{N_A}{Le} (D^2 - \alpha^2) \Theta - \left(\frac{1}{Le} (D^2 - \alpha^2) - \frac{S}{\sigma} \right) \Phi = 0 \tag{51}$$

$$W = 0, \Theta = 0 \text{ at } Z = 0 \text{ and at } Z = 1, \tag{52}$$

Where

$$D \equiv \frac{d}{dz} \quad \text{And} \quad \alpha = (l^2 + m^2)^{1/2} \tag{53}$$

thus α is a dimensionless horizontal wave number.

For neutral stability the real part of s is zero. Hence we now write $s = i\omega$, where ω is real and is a dimensionless frequency. We now employ a Galerkin-type weighted residuals method to obtain an approximate solution to the system of Eq (53)–(56). We choose as trial functions (satisfying the boundary conditions)

$$W_p = \Theta_p = \Phi = \sin p\pi z; \quad p = 1, 2, 3, \dots \tag{54}$$

write

$$W = \sum_{p=1}^N A_p W_p, \quad \Theta = \sum_{p=1}^N B_p \Theta_p, \quad \Phi = \sum_{p=1}^N C_p \Theta_p \tag{55}$$

substitute into Eq (49)–(51), and make the expressions on the left-hand sides of those equations (the residuals) orthogonal to the trial functions, thereby obtaining a system of 3N linear algebraic equations in the 3N unknowns $A_p, B_p, C_p, p = 1, 2, 3, \dots, N$. The vanishing of the determinant of coefficients produces the eigenvalue equation for the system. One can regard Ra_τ as the eigenvalue. Thus Ra_τ is found in terms of the other parameters.

5. Linear Stability Analysis

5.1 Non- Oscillatory Convection

First, we consider the case of non-oscillatory instability, when $\omega = 0$ for the first approximation we take $N = 1$. This produces the result

$$Ra_\tau = \frac{(\pi^2 + \alpha^2)^2}{\alpha^2} - (N_A + \frac{Le}{\varepsilon})Rn. \tag{56}$$

Finding the minimum as α varies results in

$$Ra_\tau = 4\pi^2 - \left(N_A + \frac{Le}{\varepsilon} \right) Rn. \tag{57}$$

with the minimum being attained at $\alpha = \pi$. We recognize that in the absence of nanoparticles we recover the well-known result that the critical Rayleigh number is equal to $4\pi^2$. Usually when one employs a single-term Galerkin approximation in this context one gets an overestimate by about 3% (e.g. 1750 instead of 1708 in the case of the standard Bénard problem) but in this case the approximation happens to give the exact result. As we have noted for a typical nanofluid Le is of order $10^2 - 10^3$ and N_A is not much greater than 10. Hence the coefficient of Rn in Eq (57) is large and negative. Thus under the approximations we have made so far we have the result that the presence of nanoparticles lower the value of the critical Rayleigh number usually by a substantial amount in the case when Rn is positive, that is when the basic nanoparticle distribution is a top-heavy one. It will be noted that in Eq (57) the parameter N_B does not appear. The instability is almost purely a phenomenon due to buoyancy coupled with the conservation of nanoparticles. It is independent of the contributions of Brownian motion and thermophoresis to the thermal energy equation. Rather, the Brownian motion and thermophoresis enter to produce their effects directly into the equation expressing the conservation of nanoparticles so that the temperature and the particle density are coupled in a particular way and that results in the thermal and concentration buoyancy effects being coupled in the same way. It is useful to emphasize this by rewriting Eq (57) in the form

$$Ra_T + (N_A + \frac{Le}{\epsilon})Rn = 4\pi^2$$

(58)

and noting that the left-hand side is the linear combination of the thermal Rayleigh number Ra_T and the concentration Rayleigh number Rn . The problem is analogous to the double-diffusive problem discussed in Section 9.1.1 of Nield and Bejan [23]. It is also analogous to the bioconvection problem discussed by Kuznetsov and Avramenko [24]. We have defined Rn in a way so that it is positive when the applied particle density increases upwards (the destabilizing situation). We note that Ra_T takes a negative value when Rn is sufficiently large. In this case the destabilizing effect of concentration is so great that the bottom of the fluid layer must be cooled relative to the top in order to produce a state of neutral stability. We emphasize that the simple expression in Eq (57) arises because the Lewis number has been assumed to be large. In order to estimate the contribution of the terms involving N_b we have investigated the two-term Galerkin results. The expression in the eigenvalue equation is complicated and it is difficult to make a statement that is simultaneously precise, simple and general. However, it is clear that the functions of N_b are of second degree. We conclude that for practical purposes Eq (58) is a good approximation.

5.2 Oscillatory Convection.

We now consider the case $\omega \neq 0$. We confine ourselves to the one-term Galerkin approximation. The eigenvalue equation now takes the form

$$Ra_T = \frac{1}{(\frac{J}{Le} + \frac{i\omega}{\sigma})\alpha^2} [(1 + \gamma_a i\omega)\alpha^2 J (\frac{J}{Le} + \frac{i\omega}{\sigma})(J + i\omega) - \frac{Rn}{\epsilon}(J + i\omega) - Rn \frac{N_A}{Le}(J)] \tag{59}$$

where for shorthand we have written

$$J = \pi^2 + \alpha^2 \tag{60}$$

6. Non-Linear Stability Analysis

For simplicity, we consider the case of two dimensional rolls, assuming all physical quantities to be independent of y . Eliminating the pressure and introducing the stream function we obtain

$$\nabla^2 \Psi + s\gamma_a \nabla^2 w + Ra \frac{\partial T}{\partial x} - Rn \frac{\partial S}{\partial x} = 0 \tag{61}$$

$$\frac{\partial T}{\partial t} + \frac{\partial \Psi}{\partial x} = \nabla^2 T + \frac{\partial(\Psi, T)}{\partial(x, z)} \tag{62}$$

$$\frac{1}{\sigma} \frac{\partial S}{\partial t} + \frac{1}{\epsilon} \frac{\partial \Psi}{\partial x} = \frac{1}{Le} \nabla^2 S + \frac{N_A}{Le} \nabla^2 T + \frac{1}{\epsilon} \frac{\partial(\Psi, S)}{\partial(x, z)} \tag{63}$$

We solve Eq 1–3 subjecting them to stress-free, isothermal, iso-nanoconcentration boundary conditions

$$\psi = \frac{\partial^2 \psi}{\partial z^2} = T = \phi = 0 \text{ at } z = 0, 1 \tag{64}$$

To perform a local non-linear stability analysis, we take the following Fourier expressions

$$\psi = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} A_{nm}(t) \sin(\max) \sin(n\pi z) \tag{65}$$

$$T = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} B_{nm}(t) \cos(\max) \sin(n\pi z) \tag{66}$$

$$\phi = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} C_{nm}(t) \cos(\max) \sin(n\pi z) \tag{67}$$

Further we take the modes (1, 1) for stream function and (0, 2) for temperature and (1,1) for nanoparticle concentration, to get

$$\psi = A_1(t) \sin(ax) \sin(\pi z) \tag{68}$$

$$T = A_2(t) \cos(ax) \sin(\pi z) + A_3(t) \sin(2\pi z) \tag{69}$$

$$S = A_4(t) \cos(ax) \sin(\pi z) + A_5(t) \sin(2\pi z) \tag{70}$$

where the amplitudes $A_1(t), A_2(t), A_3(t), A_4(t), A_5(t)$ are functions of time and are to be determined. Taking the orthogonality condition with the eigenfunctions associated with the considered minimal model we get

$$\frac{\partial A_1(t)}{\partial t} = \frac{1}{\delta^2 \gamma_a} [-\delta^2 A_1(t) + a Rn A_4(t) - a Ra A_2(t)] \tag{71}$$

$$\frac{dA_2(t)}{dt} = -[aA_1(t) + \delta^2 A_2(t) + a\pi A_1(t)A_3(t)] \tag{72}$$

$$\frac{dA_3(t)}{dt} = -4\pi^2 A_3(t) + \frac{a\pi}{2} A_1(t)A_2(t) \tag{73}$$

$$\frac{dA_4(t)}{dt} = -\frac{1}{\sigma} \left[\frac{1}{\varepsilon} aA_1(t) + \delta^2 \left[\frac{A_4(t)}{Le} + \frac{N_A}{Le} A_2(t) \right] + \frac{1}{\varepsilon} aA_1(t)A_5(t) \right] \tag{74}$$

$$\frac{dA_5(t)}{dt} = -\frac{1}{\sigma} \left[\frac{1}{Le} 4\pi^2 A_5(t) + 4\pi^2 A_3(t) \frac{N_A}{Le} - \frac{1}{\varepsilon} \frac{a\pi}{2} A_1(t)A_4(t) \right] \tag{75}$$

thus we get

$$D_1 = \frac{1}{\delta^2 \gamma_a} [-\delta^2 A_1(t) + a Rn A_4(t) - a Ra A_2(t)] \tag{76}$$

$$D_2 = -[aA_1(t) + \delta^2 A_2(t) + a\pi A_1(t)A_3(t)] \tag{77}$$

$$D_3 = -4\pi^2 A_3(t) + \frac{a\pi}{2} A_1(t)A_2(t) \tag{78}$$

$$D_4 = -\frac{1}{\sigma} \left[\frac{1}{\varepsilon} aA_1(t) + \delta^2 \left[\frac{A_4(t)}{Le} + \frac{N_A}{Le} A_2(t) \right] + \frac{1}{\varepsilon} aA_1(t)A_5(t) \right] \tag{79}$$

$$D_5 = -\frac{1}{\sigma} \left[\frac{1}{Le} 4\pi^2 A_5(t) + 4\pi^2 A_3(t) \frac{N_A}{Le} - \frac{1}{\varepsilon} \frac{a\pi}{2} A_1(t)A_4(t) \right] \tag{80}$$

$$\text{and } D_2 = D_3 = D_4 = D_5 = 0 \tag{81}$$

One may also conclude that the trajectories of the above equations will be confined to the finiteness of the ellipsoid. Thus, the effect of the parameters Rn, Le, N_A on the trajectories is to attract them to a set of measure zero or to a fixed point to say.

6.1 Heat and Nanoparticle Concentration Transport

The thermal Nusselt number $Nu_f(t)$ is defined as

$$Nu_f(t) = \frac{\text{Heat transport by (conduction + convection)}}{\text{Heat transport by conduction}} \tag{82}$$

$$= 1 + \left[\frac{\int_0^{2\pi/a} \left(\frac{\partial T}{\partial Z} \right) dx}{\int_0^{2\pi/a} \left(\frac{\partial T_B}{\partial z} \right) dx} \right]_{z=0} \tag{83}$$

Substituting expressions (45) and (73) in Eq (86) we get

$$Nu_f(t) = 1 - 2\pi A_3(t) \quad (84)$$

The nanoparticle concentration Nusselt number $Nu_\phi(t)$ is defined similar to the thermal Nusselt number. Following the procedure adopted for arriving at $Nu(t)$ one can obtain the expression for $Nu_\phi(t)$ in the form

$$Nu_\phi(t) = (1 - 2\pi A_3(t)) + N_A(1 - 2\pi A_3(t)) \quad (85)$$

7. Results and Discussion

The expressions of thermal Rayleigh number for stationary and oscillatory convections are given by equation (56) and (59) respectively.

Fig.1.a-d shows the effect of various parameters on the neutral stability curves for stationary convection for $Rn = -0.1$, $Le = 200$, $N_A = -5$, $\varepsilon = 0.9$ with variation in one of these parameters. The effect of nanoparticle concentration Rayleigh number Rn is shown in Fig. 1a. It is shown that the thermal Rayleigh number decreases with increase with increase in nanoparticle concentration Rayleigh number Rn which means that nanoparticle concentration Rayleigh number Rn destabilizes the system. It should be noted that the negative value of Rn indicates a bottom heavy case while a positive value indicates a top-heavy case. The effect of Lewis number Le on the thermal Rayleigh number is shown in Fig. 1b one can see that the thermal Rayleigh number increases with increases in Lewis number Le , indicating that the Lewis number stabilizes the system. The effect of modified diffusivity ratio N_A on the thermal Rayleigh number is shown in Fig.1c that as N_A increases Ra_T increases and hence N_A has a stabilizing effect on the system. From Fig.1d one can observe that as porosity ε increases thermal Rayleigh number decreases which means that the porosity advances the onset of convection.

Fig.2.a-f displays the variation of thermal Rayleigh number for oscillatory convection with respect to various parameters. In Fig.2a it is seen that for negative values of Rn (bottom heavy case) the thermal Rayleigh number decreases as Rn increases which will delay the onset of convection. As the Lewis number Le increases the thermal Rayleigh Number Ra_T decreases as seen in Fig.2b which imply that Lewis Number Le destabilizes the system. The modified diffusivity ratio N_A do not show any effect on the oscillatory convection (Fig.2c) from the picture 2d, one can reveal that the porosity ε destabilizes the system for oscillatory convection that is an increase in ε decreases the thermal Rayleigh number. As the thermal capacity ratio σ increases, the thermal Rayleigh number also increases as can be observed in Fig.2e, which implies that σ has a stabilizing effect on the system for oscillatory convection. In Fig.2f as thermal Rayleigh number increases Vadasz number decreases which will lead to destabilize the system.

The nonlinear analysis provides not only the onset threshold of finite amplitude motion but also the information of heat and mass transport in terms of Nusselt Nu and Sherwood Sh numbers. The Nu and Sh are computed as the function of Ra_T (thermal Rayleigh number) and the variation of these non-dimensional numbers with Ra_T for different parameter values are depicted in Fig.3a – c and 4a - c respectively. In Fig.3a – c and 4a – c it is observed that each case Sherwood number is always greater than Nusselt number and both Nusselt number and Sherwood number start with the condition state value 1 at the point of onset of steady finite amplitude convection. When Ra_T is increased beyond Ra_T there is a sharp increased in the value of both Nu and Sh . However further increase in Ra will not change Nu and Sh significantly. It is to be noted that the upper bound of Nu is 3 (similar result were obtained by Malshetty et al). It should also be noted that the upper bound of Sh is not 3 (similar results were obtained by

Bhadauria et al). The upper bound of Nu remains 3 only for both clear and nanofluid where as the upper bound for Sh for clear fluid is 3 but for nanofluid it is not fixed.

In Fig.3a and 4a we observe that as the concentration Rayleigh number Rn increases the value of Nu and Sh decreases, thus showing a decrease in the rate of heat and mass transport. Fig.3b and 4b shows that as Lewis number increases both Nu and Sh decreases which imply that increasing the Lewis number suppresses the heat and mass transport. In Fig. 3c and 4c we observe that on increasing modified diffusivity ratio N_A there is no effect on the Nusselt number where as it increases the Sherwood number (which is similar result observed by Bhadauria et al).

In Fig.5a it is observed that as Rn increases Nu decreases, thus showing a decrease in the heat transport which is similar result observed by Agrawal et al [26]. From Fig.5b we observe as Lewis number increases the Nu decreases indicating that there is retardation on heat transport. The modified diffusivity ratio enhances the heat transport as see in Fig.5c.

It is seen in Fig.6a as nanoparticle concentration Rayleigh number Rn increases the Sherwood number (concentration Nusselt number) decreases, which implies the suppress of mass transport. The mass transport is enhanced for Lewis number Ln and modified diffusivity ratio N_A as see in Fig.6b and 6c respectively.

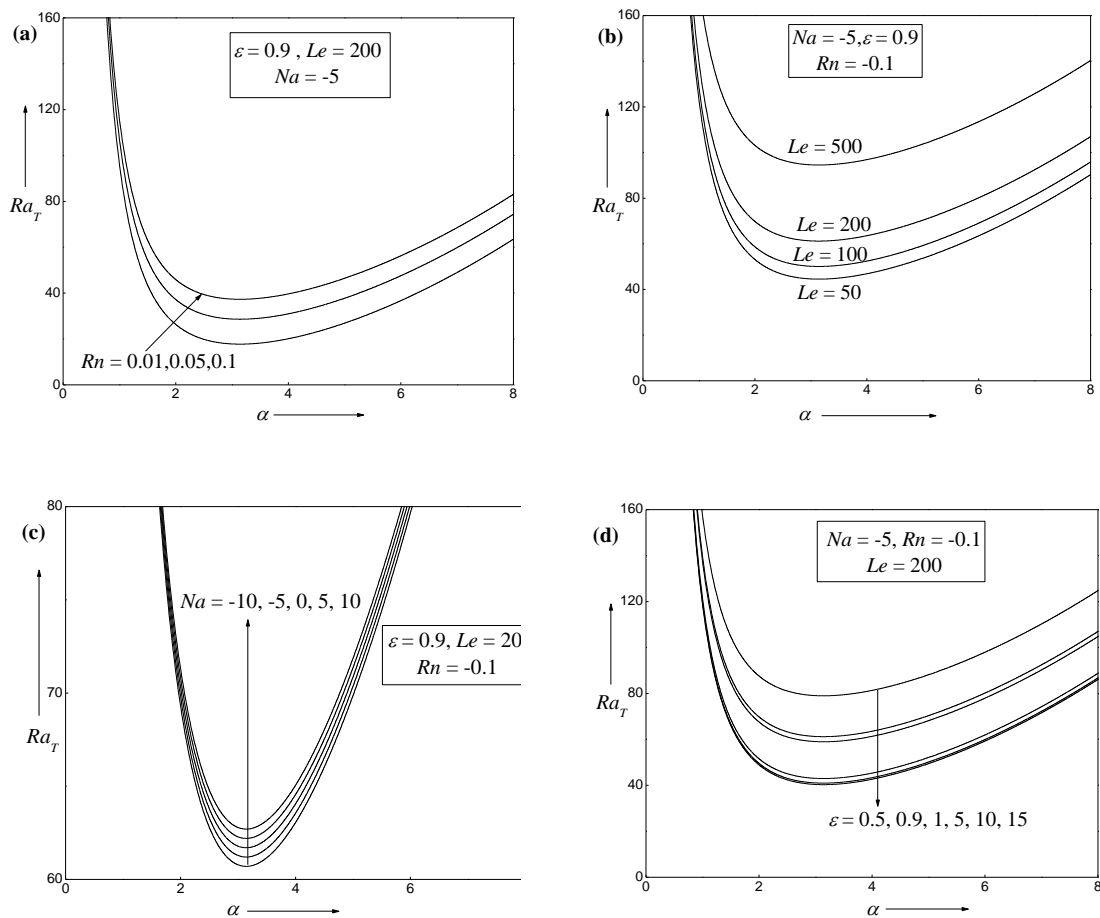


Figure 1. Neutral curve on stationary convection for different values of (a) nanoparticle concentration Rayleigh number Rn (b) Lewis number Le (c) Modified diffusivity ratio N_A (d) Porosity ϵ

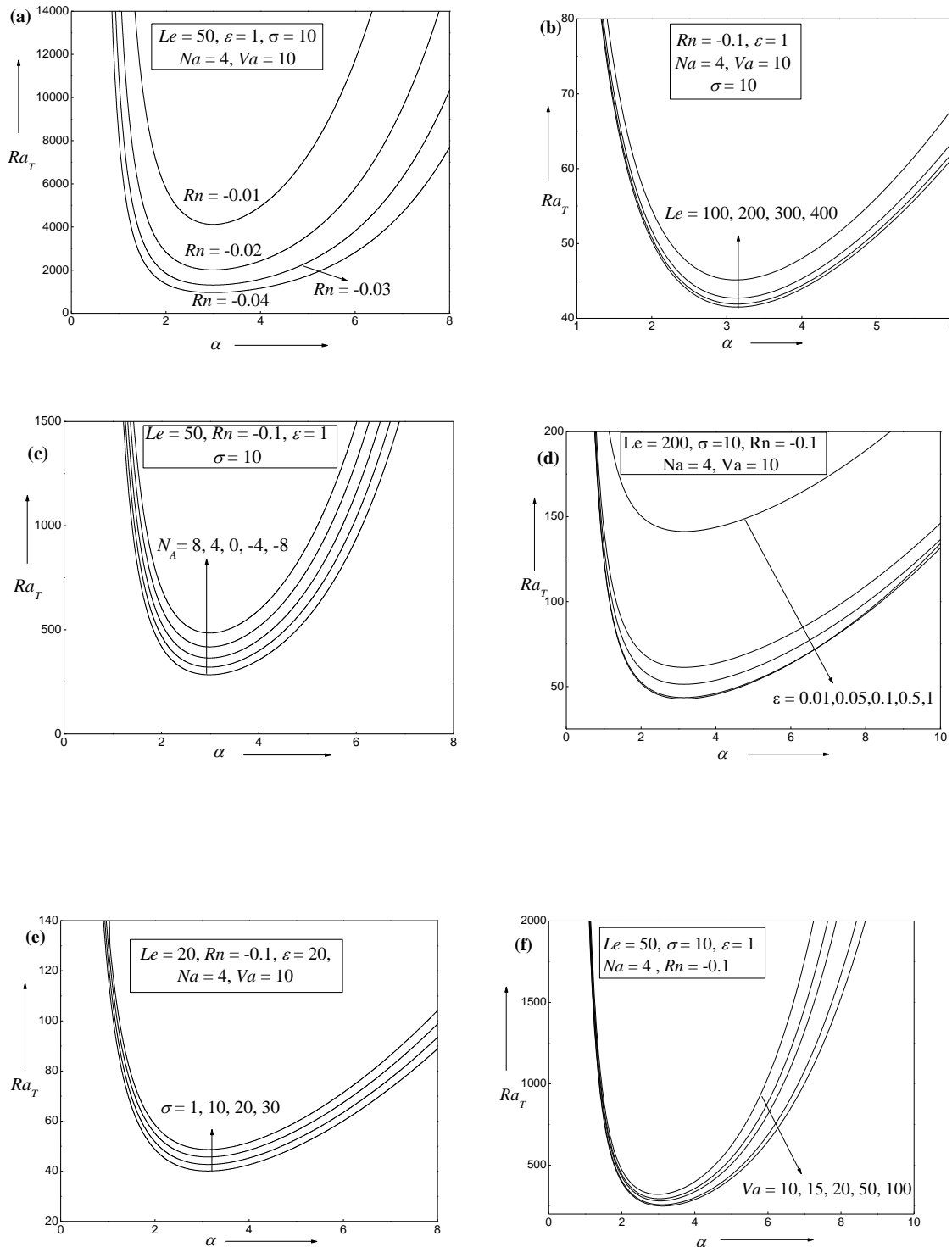


Figure 2. Neutral curve on oscillatory convection for different values of **(a)** nanoparticle concentration Rayleigh number Rn **(b)** Lewis number Le **(c)** Modified diffusivity ratio N_A **(d)** Porosity ϵ **(e)** Thermal capacity ratio σ **(f)** Vadasz number Va

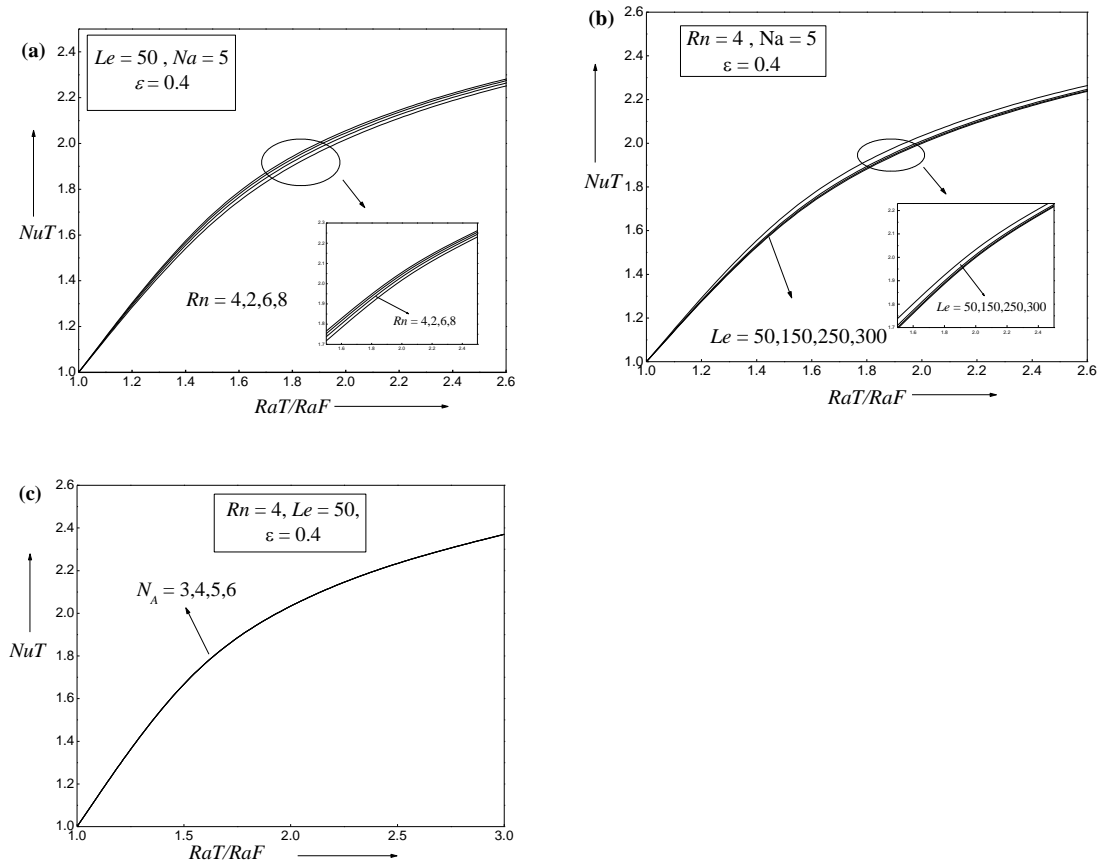
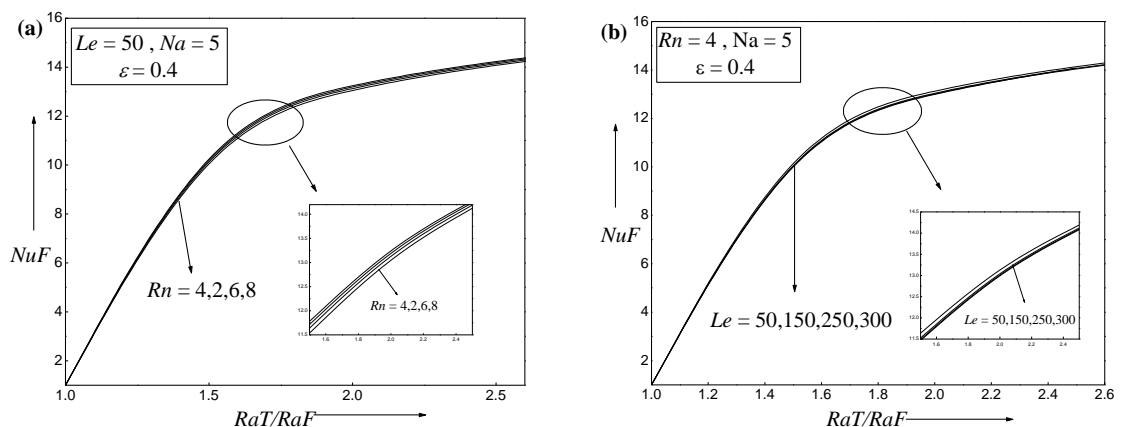


Figure 3. Variation of Nusselt number Nu with critical Rayleigh number for different values of **(a)** Nanoparticle concentration Rayleigh number Rn **(b)** Lewis number Le **(c)** Modified diffusivity ratio N_A



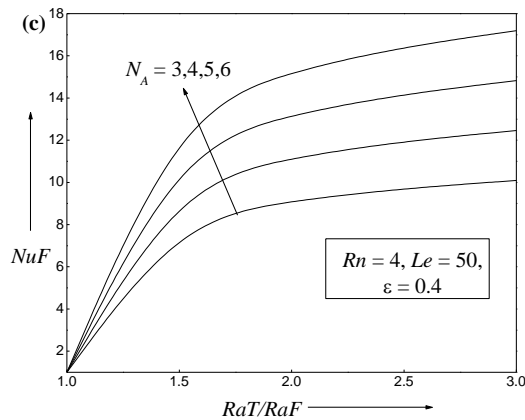


Figure 4. Variation of Sherwood Number Sh with critical Rayleigh number for different values of (a) Nanoparticle concentration Rayleigh number Rn (b) Lewis number Le (c) Modified diffusivity ratio N_A

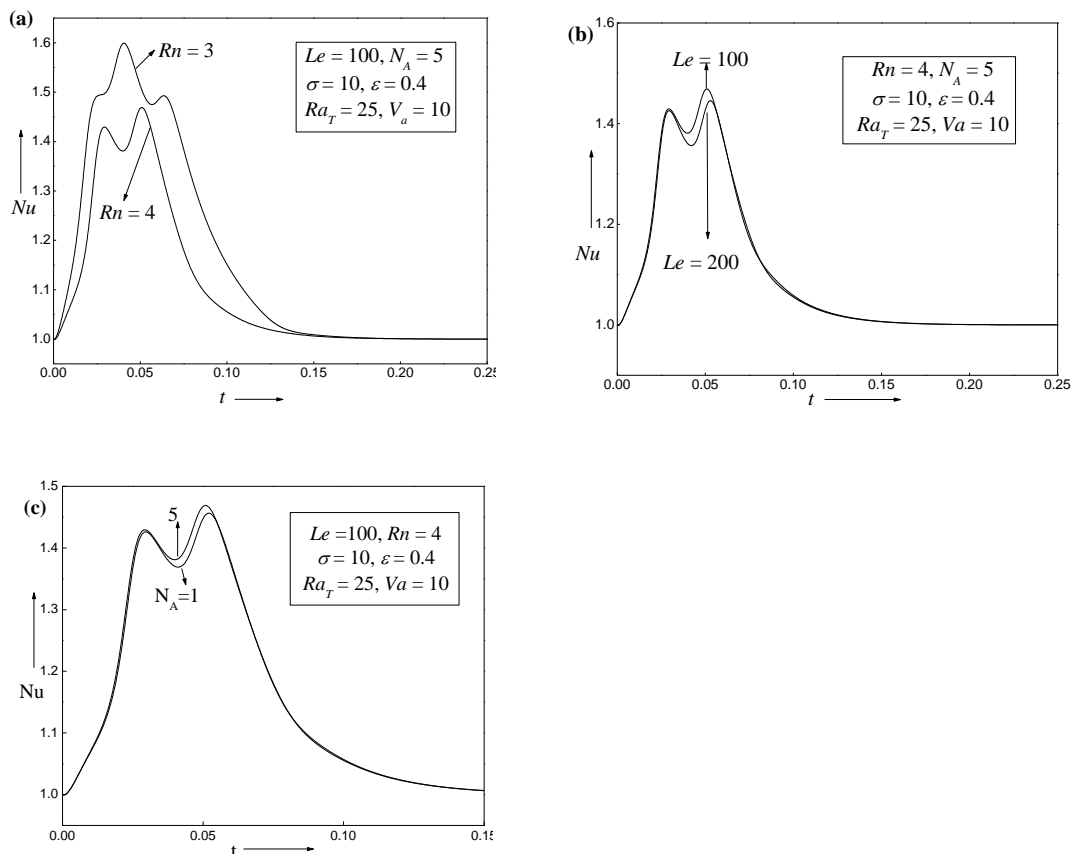


Figure 5. Variation of Nusselt number Nu with Rayleigh number for different values of (a) Nanoparticle concentration Rayleigh number Rn (b) Lewis number Le (c) Modified diffusivity ratio N_A

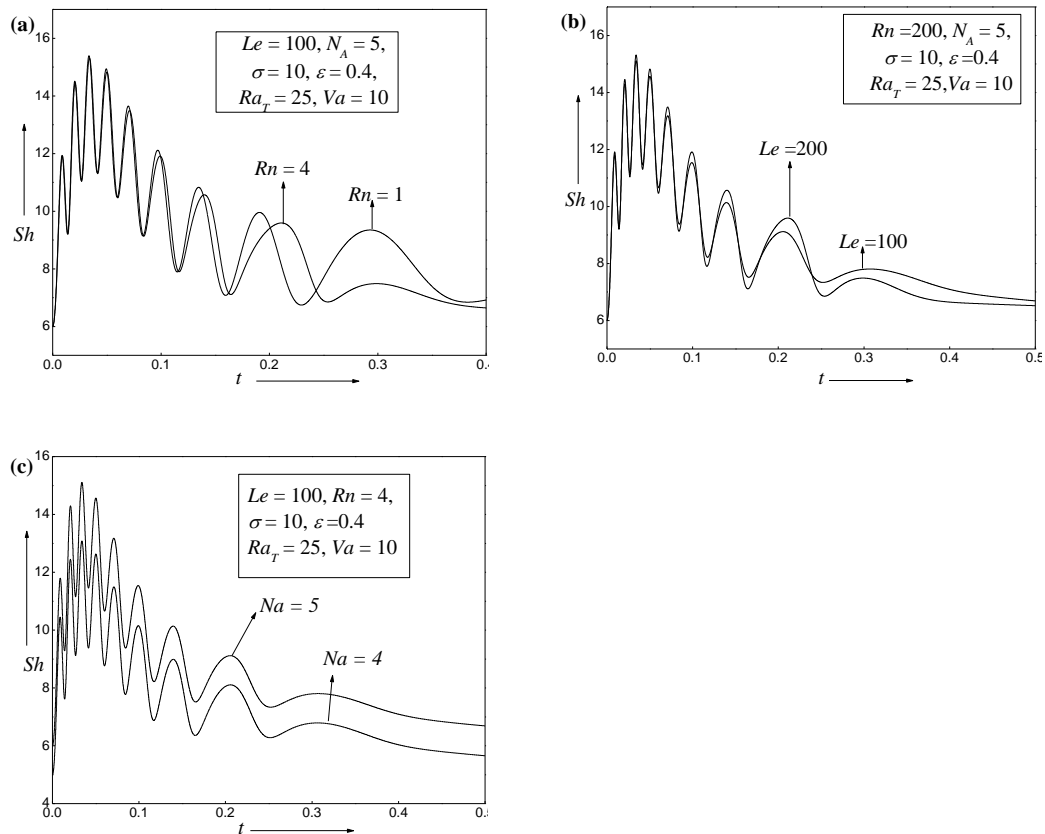


Figure 6. Variation of Sherwood Number Sh with Rayleigh number for different values of (a) Nanoparticle concentration Rayleigh number Rn (b) Lewis number Le (c) Modified diffusivity ratio N_A

Conclusion

We consider linear stability analysis in a horizontal porous medium saturated by a nanofluid, heated from below and cooled from above, using Darcy model which incorporates the effect of Brownian motion along with thermophoresis. Linear analysis has been made using normal mode technique. However for weakly nonlinear analysis truncated Fourier series representation having only two terms is considered. We draw the following conclusions.

1. For stationary convection Lewis number Le modified diffusivity ratio N_A has a stabilizing effect while nanoparticle concentration Rayleigh number Rn and porosity destabilize the system
2. For oscillatory convection thermal capacity ratio σ stabilizes the system where as nanoparticle concentration Rayleigh number Rn , Lewis number Le and porosity, vadasz number Va destabilizes the system.
3. For steady finite amplitude motions, the heat and mass transport decreases with increase in the values of nanoparticle concentration Rayleigh number Rn , Lewis number Le , the mass transport increases with increases in modified diffusivity ratio
4. The transient Nusselt number and Sherwood number increases with increases in Lewis number Le and modified diffusivity ratio N_A and decreases with nanoparticle concentration Rayleigh number Rn

5. The effect of time on transient Nusselt number and Sherwood number is found to be oscillatory when t is small. However when t becomes very large both the transient Nusselt and Sherwood value approaches to the steady and Sherwood value approaches to the steady value.

Nomenclature

c	Nanofluid specific heat at constant pressure
c_p	Specific heat of the nanoparticle material
$(\rho c)_m$	Effective heat capacity of the porous material
d_p	Nanoparticle diameter
D_B	Brownian diffusion coefficient (m^2/s), given by Eq.(4)
D_T	Thermophoretic diffusion coefficient (m^2/s), given by Eq.(8)
h_p	Specific enthalpy of the nanoparticle material
H	Dimensional layer depth (m)
\mathbf{j}_p	Diffusion mass flux for the nanoparticles, given by Eq.(7)
$\mathbf{j}_{p,T}$	Thermophoretic diffusion, given by Eq.(7)
k	Thermal conductivity of the nanofluid (W/m K)
k_B	Boltzmann's constant
k_m	Overall thermal conductivity of the porous medium saturated by the nanofluid
k_p	Thermal conductivity of the particle material
K	Permeability (m^2)
Le	Lewis number
N_A	Modified diffusivity ratio
N_B	Modified particle-density increment
p^*	Pressure (Pa)
p	Dimensionless pressure, $p^* K / \mu \alpha_f$
\mathbf{q}	Energy flux relative to a frame moving with the nanofluid velocity \mathbf{v}
Ra_T	Thermal Rayleigh- Darcy number
Rm	Basic-density Rayleigh number
Rn	Concentration Rayleigh number
Va	Vadasz number
t^*	Time (s)
t	Dimensionless time, $t^* \alpha_f / H^2$
T^*	Nanofluid temperature (K)
T	Dimensionless temperature, $\frac{T^* - T_c^*}{T_h^* - T_c^*}$
T_c^*	Temperature at the upper wall (K)
T_h^*	Temperature at the lower wall (K)
\mathbf{v}	Nanofluid velocity (m/s)
\mathbf{v}_D	Darcy velocity $\varepsilon \mathbf{v}$
\mathbf{v}_D^*	Dimensional Darcy velocity
\mathbf{V}_T	Thermophoretic velocity

- (u, v, w) Dimensionless Darcy velocity components $(u^*, v^*, w^*)H/\alpha_m$ (m/s)
 (x, y, z) Dimensionless Cartesian coordinate $(x^*, y^*, z^*)/H$; z is the vertically upward
 (x^*, y^*, z^*) Cartesian coordinates

Greek symbols

- α_m Thermal diffusivity of the porous medium
 $\tilde{\beta}$ Proportionality factor, given by Eq.(6)
 μ Viscosity of the fluid
 $\tilde{\mu}$ Effective viscosity of the porous medium
 ρ Fluid density
 ρ_p Nanoparticle mass density
 σ Thermal capacity ratio
 ϕ^* Nanoparticle volume fraction
 ϕ Relative nanoparticle volume fraction, $\frac{\phi^* - \phi_0^*}{\phi_1^* - \phi_0^*}$

Superscripts

- * Dimensional variable
' Perturbed variable

Subscripts

- b Basic solution
f Fluid
p Particle

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